Contents

1. Textbook

- Fundamental of Physics, Halliday, Resnick, Walker
- Intro to Electrodynamics, Griffiths
- Electricity and Magnetism, Purcell and Morin

2. Electrostatics I - Coulomb's Law

Object "participate" in electric forces when they have electric charge. The electric force between two point charges is given by Coulomb's law:

- electric charge can be $(+)$ or $(-)$
  - electric force can be attractive or repulsive

0	0	none
+	+	repel
-	-	repel
+	-	attract

2.1. Coulomb's Law

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2} \cdot \hat{r}$$

- ε_0 - "permittivity of free space" $\approx 8.85 \times 10^{-12} C^2 N^{-1} m^{-2}$
- q - electric charge, $[q] = C$
- r - the distance between the two charges
- $\hat{r}$ - unit vector that points between charges

2.2. The Principle of Superposition

This principle tells us the interaction between 2 charges is unaffected by the presence of any other charges.

$$\vec{F}_{\text{total},q} = \sum_i \vec{F}_{q,q_i}$$

3. Electrostatics II - The Electric Field

Imagine that N static charges q_i that are fixed in space. We do have a "test charge" Q that we are interested in. We could compute the force on a test charge.

$$\vec{F} = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{Q q_i}{r_i^2} \hat{r_\imath}$$
$$= Q \sum_{i=1}^{N} \frac{1}{4\pi\varepsilon_0} \frac{q_i}{r_i^2} \hat{r_\imath}$$

We define the electric field $\vec{E}$ for discrete charges as the force per unit charge:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{N} \frac{q_i}{r_i^2} \hat{r_\imath}$$

For continuous charge distributions, we can write the electric field as an integral:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\mathrm{d}q}{r^2} \hat{r}$$

- Along a line: $\mathrm{d}q = \lambda \, \mathrm{d}l$
  - λ is the "linear charge density"
- Along a surface: $\mathrm{d}q = \sigma \, \mathrm{d}A$
  - σ is the "surface charge density"
- Along a volume: $\mathrm{d}q = \rho \, \mathrm{d}V$
  - ρ is the "volume charge density"

4. Electrostatics III - Gauss's Law

- New concept - flux
  - flux is a measure of how much a vector field "flows" through a surface

Define the flux as:

$$\Phi := \int \mathrm{d}\vec{A} \cdot \vec{v}$$

Consider a closed surface, the net flux ≡ difference between the flux entering and leaving.

4.1. Gauss's Law

$$\oiint \mathrm{d}\vec{A} \cdot \vec{E} = \frac{Q_{\text{encl.}}}{\varepsilon_0}$$

- always true
  - only helpful for symmetric layouts - spherical, cylindrical, planar

5. Electrostatics IV - Conductors & Insulators

- Conductors are materials where charges are free to flow The charges rearrange themselves in response to the presence of an external electric field. Inside a conductor, $\vec{E}_{\text{net}} = 0$.
- Insulators are materials where charges cannot flow freely.

6. Electrostatics V - Electric Potential

Recall the work done by a force $W = \int \mathrm{d}\vec{x} \cdot F$

- conservative force W is "path-independent" $\Rightarrow \oint \mathrm{d}\vec{x} \cdot F = 0$
  - conservative force **conserves** energy

For conservative forces:

$$\vec{F} = -\nabla \vec{U}$$

where $\vec{U}$ is the potential energy.

We're interested in forces on arbitrary charges, q, $\vec{F} = q\vec{E}$

We can write the electric field as the gradient of a scalar potential:

$$\vec{E} = -\nabla V$$

where V is the electric potential.

$$U_{\text{elec}} = qV$$

1. V is the potential energy per unit charge
   - it's the work needed per charge to move a test charge from a reference point (usually ∞) to a specific point in the field
   - unit $[v] = \frac{J}{C} \equiv 1 \text{ volt}$
2. The electric potential as a contour map of the electric field
   - We can define "equipotential surfaces" where the potential is constant
   - The electric field points perpendicular to these equipotential surfaces, pointing from high to low

Another nice property: the potential obeys the principle of superposition:

$$V_{\text{net}} = \sum_i V_i$$

6.1. How do we compute the potential?

$$V(\vec{r}) := -\int_O^{\vec{r}} \mathrm{d}\vec{l} \cdot \vec{E}(\vec{r}')$$

where O is an agreed upon reference point where $V(O) = 0$. (Exception: doesn't work for infinite charge distributions)

If we don't know $\vec{E}$, but do know the charge distribution:

$$\text{pt. charges} : V = \frac{1}{4\pi\varepsilon_0} \sum_i \frac{q_i}{\|\vec{r}_i\|}$$

$$\text{charge distr.} : V = \frac{1}{4\pi\varepsilon_0} \int \frac{\mathrm{d}q}{\|\vec{r}\|}$$

- line charge: $\mathrm{d}q = \lambda \, \mathrm{d}l$
- surface charge: $\mathrm{d}q = \sigma \, \mathrm{d}A$
- volume charge: $\mathrm{d}q = \rho \, \mathrm{d}\tau$

7. Circuits I - Ohm's Law

$$|\Delta V| = IR$$

- $|\Delta V|$ - voltage
- I - current, charge / time
- R - resistance

$$|\Delta V| = El$$

$$E = \frac{|\Delta V|}{l}$$

$$v_D = a\tau = \frac{F}{m} \cdot \tau = \frac{Ee}{m} \cdot \tau = \frac{|\Delta V|e}{lm}\tau$$

Let v_T be the thermal velocity of electrons, and λ be the "mean free path" - average distance objects travel before colliding, we have:

$$\tau = \frac{\lambda}{v_T}$$

$$v_D = \frac{|\Delta V|e}{lm} \frac{\lambda}{v_T}$$

The current I is given by product of number of charges and v_D for each charge:

$$\begin{aligned} I &= n \cdot v_D \\ &= NA \cdot e \cdot \frac{|\Delta V|e}{lm} \frac{\lambda}{v_T} \\ &= |\Delta V| \left(\frac{e^2 N \lambda}{m v_T} \right) \frac{A}{l} \end{aligned}$$

Comparing to Ohm's law, we can identify the resistance:

$$|\Delta V| = IR$$

$$R = \frac{m v_T}{e^2 N \lambda} \frac{l}{A}$$

Usually, we write the resistance in terms of area, length, and conductivity σ:

$$R = \frac{l}{\sigma A}$$

Therefore, we can have σ defined as:

$$\sigma = \frac{e^2 N \lambda}{m v_T}$$

As a result from Thermodynamics, we have:

$$\frac{1}{2} m v_T^2 = \frac{3}{2} k_B T$$

$$v_T = \sqrt{\frac{3 k_B T}{m}}$$

$$R = \frac{\sqrt{3 k_B T m}}{e^2 N \lambda} \frac{l}{A}$$

which shows that the absolute temperature T must be 0 for the material to be superconducting.

7.1. Vocab Terms

- Circuit: a closed loop consisting of circuit elements
- Circuit element: anything that adds or removes energy from the circuit
- Current: moving electric charge, $I = \frac{\mathrm{d}q}{\mathrm{d}t}$, measured in Amperes (A)
- Wires: equipotential surfaces connecting circuit elements
- Series circuit: only 1 path for charge to flow
- Parallel circuit: multiple paths for charge to flow

7.2. Circuit Rules

1. Need ΔV for charge to flow
2. Need a path for charge to flow
3. Around any closed loop, $\Delta V = 0$ (Kirchhoff's loop rule, energy conservation)
4. At any junction, $I_{\text{in}} = I_{\text{out}}$ (Kirchhoff's junction rule, charge conservation)

5. Across any resistor, $\Delta V = IR$ (Ohm's law)

8. Circuit Analysis

Goal: Predict ΔV across and I through resistors.

$\rightarrow$ Crucial step: Can we "simplify" our model of a circuit?

For resistors in series:

$$R_{\text{eq}} = \sum_n R_n$$

For resistors in parallel:

$$\frac{1}{R_{\text{eq}}} = \sum_n \frac{1}{R_n}$$

9. Capacitors

- Capacitors are circuit elements that stores energy in electric fields by letting charges accumulate on closely spaced plates.
- Definition:

  $$C = \frac{Q}{V}$$

  where Q is charge stored on plates and V is potential difference across plates.

  Measured in Farads (F): $1F = 1\frac{C}{V}$.
- Capacitors in series:

  $$\frac{1}{C_{\text{eq}}} = \sum_n = \frac{1}{C_n}$$

- Capacitors in parallel:

  $$C_{\text{eq}} = \sum_n C_n$$

10. Time Dependence in Capacitors

$$Q(t) = CV(t)$$
$$I(t) := \frac{\mathrm{d}Q}{\mathrm{d}t} \equiv \dot{Q} = C\frac{\mathrm{d}V}{\mathrm{d}t}$$

This gives the analogous of Ohm's law for capacitors:

$$\boxed{I = C\frac{\mathrm{d}V}{\mathrm{d}t}}$$

We can also get the inverse relationship:

$$I(t) = C\frac{\mathrm{d}V}{\mathrm{d}t}$$
$$\mathrm{d}V = \frac{1}{C}I(t)\,\mathrm{d}t$$
$$\int \mathrm{d}V = \int \frac{1}{C}I(t)\,\mathrm{d}t$$
$$V(t) - V(0) = \frac{1}{C}\int I(t)\,\mathrm{d}t$$

This gives the relationship between voltage and current for a capacitor:

$$\boxed{V(t) = V(0) + \frac{1}{C}\int I(t)\,\mathrm{d}t}$$

11. RC Circuits

How do we analyze circuits?

1. Combine resistors and capacitors into equivalent resistors and capacitors
2. Use KLR and KJR to write equations for ΔV and I
3. Use $I - V$ relationships to turn these into diff. eq.s or algebraic eq.s
4. Do the math

11.1. Charging a Capacitor

What do we expect to happen as time passes?

$$Q(t) : \text{at } t = 0 \text{ , } Q = 0$$
$$\lim_{t \to \infty} Q = Q_{\max}$$

$$I(t) : \text{at } t = 0 \text{ , } I = I_{\max}$$
$$\lim_{t \to \infty} I = 0$$

$$V_R(t) : \text{at } t = 0 \text{ , } V_R = 0$$
$$\lim_{t \to \infty} V_R = 0$$

$$V_C(t) : \text{at } t = 0 \text{ , } V_C = 0$$
$$\lim_{t \to \infty} V_C = \Delta V_B = V_{\max}$$

We have the equation:

$$\Delta V_B - V_R - V_C = 0$$
$$\Delta V_B - IR - \frac{Q}{C} = 0$$
$$\Delta V_B - R\frac{\mathrm{d}Q}{\mathrm{d}t} - \frac{Q}{C} = 0$$

After solving the diff. eq.s, we get:

$$Q(t) = \Delta V_B C \left(1 - e^{-\frac{t}{RC}}\right)$$

We let $\tau = RC$ the “time constant” of the circuit, and let $Q_{\max} = \Delta V_B C$, which gives:

$$Q(t) = Q_{\max}\left(1 - e^{-t/\tau}\right)$$

Consequently, we have:

$$I(t) = \frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{Q_{\max}}{\tau} e^{-t/\tau} = \frac{\Delta V C}{RC} e^{-t/\tau} = \frac{\Delta V}{R} e^{-t/\tau}$$

$$I(0) = I_0 = \frac{\Delta V_B}{R}$$

and similarly:

$$\begin{aligned} V_R(t) &= I(t)R = I_0 R e^{-t/\tau} = \Delta V_B e^{-t/\tau} \\ V_C(t) &= V(0) + \frac{1}{C}\int_0^t \mathrm{d}t'\, I(t') \\ &= \Delta V_B \left(1 - e^{-t/\tau}\right) \end{aligned}$$

11.2. Discharging a Capacitor

$$Q(t) : \text{at } t = 0 \text{ , } Q = Q_{\text{max}}$$
$$\lim_{t\to\infty} Q = 0$$

$$I(t) : \text{at } t = 0 \text{ , } I = I_{\text{max}}$$
$$\lim_{t\to\infty} I = 0$$

$$V_R(t) : \text{at } t = 0 \text{ , } V_R = V_{\text{max}}$$
$$\lim_{t\to\infty} V_R = 0$$

$$V_C(t) : \text{at } t = 0 \text{ , } V_C = V_{\text{max}}$$
$$\lim_{t\to\infty} V_C = 0$$

12. Intro to Magnetostatics

- We know magnets have north and south poles (analogous to (+) and (-) charges)
  - like poles repel, opposite poles attract
- Weird because we always have north and south poles together
  - there are no magnetic monopoles

12.1. "Gauss's Law for Magnetism":

$$\oiint \mathrm{d}\vec{A} \cdot \vec{B} = 0$$

- $\vec{B}$ denotes the magnetic field
- magnetic field is measured in Tesla (T): $1T = 1\frac{N}{C\frac{m}{s}}$

Particles feel forces from magnetic fields

$$\vec{F}_{\vec{B}} = q\vec{v} \times \vec{B}$$

- $\vec{v}$ - velocity of the charge
- magnitude:

$$\left\|\vec{F}_{\vec{B}}\right\| = q\ \|\vec{v}\|\left\|\vec{B}\right\| \sin \theta_{\vec{v}\vec{B}}$$

12.2. Right Hand Rule

We can determine the direction of the force using the right hand rule:

- pointer finger in direction of $\vec{v}$
- middle finger in direction of $\vec{B}$
- thumb points in direction of $\vec{F}$

Notation for in/out of page:

- $\odot \rightarrow$ out of the page
- $\otimes \rightarrow$ into the page

12.3. Lorentz Force Law

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

Lorentz force law combines the electric and magnetic forces on a charge. It's a generalization of Coulomb's law (for electric field) and the magnetic force law (for magnetic field).

13. Magnetostatics II - Biot-Savart Law

- Moving charges react to $\vec{B}$ fields
- Moving charges also create $\vec{B}$ fields
- Look at the $\vec{B}$ field from steady currents ($I(t) = I_0$)

The Biot-Savart Law is an empirical law that lets us find $\vec{B}$ at any distance away from a wire carrying a steady current:

$$\mathrm{d}\vec{B} = \frac{\mu_0}{4\pi} \frac{I \,\mathrm{d}\vec{l} \times \hat{r}}{r^2}$$

- μ_0 - "permeability of free space", how easily $\vec{B}$ fields can move through space
  - $\mu_0 \approx 4\pi \times 10^{-7} \frac{N}{A^2}$
- $\mathrm{d}\vec{l}$ - differential length element of the wire, points in direction of current
- r - distance from $\mathrm{d}\vec{l}$ to the point of interest
- $\hat{r}$ - unit vector pointing from $\mathrm{d}\vec{l}$ to the point of interest

Strategy:
1. Use RHR to determine direction of $\mathrm{d}\vec{B} \Rightarrow$ focus on just the mag. of $\mathrm{d}\vec{B}$
2. Clearly define $\mathrm{d}\vec{l}$, r, $\hat{r}$
3. Pray for an easy integral

14. Magnetostatics III - Ampère's Law

Before we generalize BSL, let's first generalize our notion of current:

- $\vec{I}$ - linear current
- $\vec{K}$ - surface current density
- $\vec{J}$ - volume current density

$$\sum_i q_i v_i \sim \int_{\text{line}} \mathrm{d}l \, \vec{I} \sim \int_{\text{surface}} \mathrm{d}A \, \vec{K} \sim \int_{\text{volume}} \mathrm{d}V \, \vec{J}$$

We found that $\vec{B}$-fields wrap around wires, and, for infinite straight wires, we have:

$$\vec{B}(s) = \frac{\mu_0 I}{2\pi s}\,\hat{\varphi}$$

If we integrate $\vec{B}$ around the circular path of radius s:

$$\oint \mathrm{d}\vec{l} \cdot \vec{B} = \oint \frac{\mu_0 I}{2\pi s}\,\mathrm{d}l = \frac{\mu_0 I}{2\pi s} \oint \mathrm{d}l = \mu_0 I$$

In cylindrical coordinates, we can write:

$$\mathrm{d}\vec{l} = \mathrm{d}s\,\hat{s} + s\,\mathrm{d}\varphi\,\hat{\varphi} + \mathrm{d}z\,\hat{z}$$

Thus, we have:

$$\oint \mathrm{d}\vec{l} \cdot \vec{B} = \oint s\,\mathrm{d}\varphi \cdot \vec{B} = \mu_0 I$$

The path we take can be arbitrary. Therefore, we have Ampère's Law:

$$\boxed{\oint_C \mathrm{d}\vec{l} \cdot \vec{B} = \mu_0 I}$$

If we have a bundle of straight wires, each contributing $\mu_0 I$:

$$\oint_C \mathrm{d}\vec{l} \cdot \vec{B} = \mu_0 I_{\mathrm{encl}}$$

where:

$$I_{\mathrm{encl}} = \iint_S \mathrm{d}A \cdot \vec{J} \quad \text{where } C = \partial S$$

Our "derivation" assumed straight wires with infinite length, but Ampère's law holds for any wire with a steady current.

AL is always true, but only sometimes useful. Looking for symmetric current distributions:

- Infinite long wire
- Infinite plane of current
- Infinite solenoid ($l \gg r$)
- Toroid

14.1. AL Strategy:

1. Identify the symmetry. If not, BSL
2. Determine the direction of $\vec{B}$ using RHR
3. Choose a loop (circle, rectangle, etc.) over wires
4. Calculate I_{encl}
5. Calculate $\oint \vec{B} \cdot \mathrm{d}\vec{l}$
6. Solve for $\vec{B}$

15. Summary

For Magnetostatics:

$$\oint \mathrm{d}\vec{l} \cdot \vec{B} = \mu_0 I_{\text{encl}} \qquad \oiint \mathrm{d}\vec{A} \cdot \vec{B} = 0$$

For Electrostatics:

$$\oint \mathrm{d}\vec{l} \cdot \vec{E} = 0 \qquad \oiint \mathrm{d}\vec{A} \cdot \vec{E} = \frac{Q_{\text{encl}}}{\varepsilon_0}$$

16. Electrodynamics I - Faraday’s Law

1831 - Michael Faraday conducted 3 experiments:
1. Poll a loop through an external $\vec{B}$ field - induced some current
2. Move a magnet to stationary loop - induced some current
3. Leave loop and magnet stationary, but make $\vec{B}$ stronger - induced some current

Faraday’s Law: A changing magnetic field induces an electric field.

$$\mathcal{E} = \oint_C \mathrm{d}\vec{l} \cdot \vec{E} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}$$

- $\mathcal{E}$ - "electromotive force", "emf" (this is just a voltage)
- C - any arbitrary closed loop
- Φ_B - the magnetic flux through the loop

$$\Phi_B = \iint_S \mathrm{d}\vec{A} \cdot \vec{B}$$

Lenz's Law: Nature abhors a change in flux.

When Φ_B through a closed loop changes, nature induced a current to create a magnetic field to counteract that change in flux.

16.1. Faraday's Law Problem Solving Strategy

1. $|\mathcal{E}|$ for Faraday's Law
2. Direction of current from Lenz's Law

17. Electrodynamics II - Ampère-Maxwell Law + Maxwell's Equations

Ampère's Law is not always with the static charge. We can correct it with it by adding a term for the "displacement current":

Traditional derivation considers a capacitor with a changing electric field between the plates.

$$\oint_C \mathrm{d}\vec{l} \cdot \vec{B} = \mu_0 I_{\text{encl}} + \mu_0 \varepsilon_0 \frac{\partial \Phi_E}{\partial t}$$

$$\oiint_S \mathrm{d}\vec{A} \cdot \vec{E} = \frac{Q_{\text{encl}}}{\varepsilon_0} = \iiint_V \mathrm{d}V \left(\nabla \cdot \vec{E}\right) \Rightarrow \nabla \cdot \vec{E} = \frac{Q}{\varepsilon_0}$$

$$\oiint_S \mathrm{d}\vec{A} \cdot \vec{B} = 0 = \iiint_V \mathrm{d}V \left(\nabla \cdot \vec{B}\right) \Rightarrow \nabla \cdot \vec{B} = 0$$

$$\oint_C \mathrm{d}\vec{l} \cdot \vec{E} = -\frac{\mathrm{d}\Psi_B}{\mathrm{d}t} = \iint_A \mathrm{d}\vec{A} \cdot \left(\nabla \times \vec{E}\right) \Rightarrow \nabla \times \vec{E} = -\frac{\mathrm{d}\vec{B}}{\mathrm{d}t}$$

$$\oint_C \mathrm{d}\vec{l} \cdot \vec{B} = \mu_0 \iint_S \mathrm{d}\vec{A} \cdot \vec{J} \Rightarrow \nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\mathrm{d}\vec{E}}{\mathrm{d}t}$$

Continuity equation: The amount of electric charge in any volume in space can only change by the amount of current flowing through its boundary:

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

17.1. Modern Derivation of Ampère-Maxwell Law

$$0 = \nabla \cdot \left(\nabla \times \vec{B}\right) = \mu_0 \nabla \cdot \vec{J} \neq 0$$

We get a conflict here. Therefore, we consider adding another term be be consistent with $\nabla \cdot \left(\nabla \times \vec{F}\right) = 0$:

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} = -\varepsilon_0 \frac{\partial}{\partial t}\left(\nabla \cdot \vec{E}\right) = -\nabla \cdot \left(\varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

which gives us:

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

18. Conclusion

18.1. Maxwell's Equations

Integral form:

$$\oiint \mathrm{d}\vec{A} \cdot \vec{E} = \frac{Q_{\text{encl}}}{\varepsilon_0} \qquad \oint \mathrm{d}\vec{l} \cdot \vec{B} = \mu_0 I_{\text{encl}} + \mu_0 \varepsilon_0 \frac{\partial \Phi_E}{\partial t}$$

$$\oiint \mathrm{d}\vec{A} \cdot \vec{B} = 0 \qquad \oint \mathrm{d}\vec{l} \cdot \vec{E} = -\frac{\mathrm{d}\Phi_B}{\mathrm{d}t}$$

Lorentz force law:

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

Differential form:

$$\begin{aligned} \boldsymbol{\nabla} \cdot \vec{E} &= \frac{\rho}{\varepsilon_0} \\ \boldsymbol{\nabla} \times \vec{B} &= \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \\ \boldsymbol{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \boldsymbol{\nabla} \cdot \vec{B} &= 0 \end{aligned}$$