1. Introduction

1.1. Incentives behind Relativistic Electrodynamics

You might have tried to play with electric and magnetic fields under different reference frames, and found that Maxwell's equations don't seem compatible with classical Galilean transforms.

For instance, consider a rested point charge q. It's electric field can be easily calculated using Gauss's law, and is given by:

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{\|\boldsymbol{r}\|^2}\hat{\boldsymbol{r}}$$

and there's no magnetic field.

However, consider another reference frame moving at a constant velocity $\boldsymbol{v} = v\hat{\boldsymbol{x}}$. Under classical mechanics, the electric field remains the same, but we also have a magnetic field given by Ampère's law:

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{q\boldsymbol{v}\times\hat{\boldsymbol{r}}}{\|\boldsymbol{r}\|^2}$$

Obviously, if we place a test charge in this field, it will experience different forces in the two reference frames, which isn't physically reasonable as we shouldn't cause different physical phenomena by just changing our perspective.

Therefore, we are motivated to consider electrodynamics under a new perspective, and that's where special relativity comes in.

1.2. Recap of Special Relativity

Recall that we follow two important postulates in special relativity:

1. Laws of physics are same under every inertial reference frame.
2. Speed of light in vacuum remains constant.

Through a series of thought experiments, we can derive the Lorentz transforms, which relate coordinates of events under different inertial reference frames.

Before listing all the equations, let's first establish some notations. We will use $\mathcal{S}$ to denote a reference frame, and the coordinates of an event in this reference frame will be denoted by $\boldsymbol{x} = (t, x, y, z)$. Accordingly, in another reference frame $\mathcal{S}'$, the coordinates of the same event will be denoted by $\boldsymbol{x}' = (t', x', y', z')$.

Suppose $\mathcal{S}'$ is moving at a constant velocity $\boldsymbol{v} = v\hat{x}$ relative to $\mathcal{S}$. We will define the Lorentz factor γ as:

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}$$

Then, we can express the Lorentz transforms as:

- $t' = \gamma(t - vx/c^2)$
- $x' = \gamma(x - vt)$
- $y' = y$
- $z' = z$

Now, consider a particle moving at a velocity $\boldsymbol{u} = \left(u_x, u_y, u_z\right)$ in $\mathcal{S}$. We can derive the velocity transforms as:

$$\boldsymbol{u}' = \frac{\boldsymbol{v} - \boldsymbol{u}}{1 - (\boldsymbol{v} \cdot \boldsymbol{u})/c^2} \tag{1}$$

This is known as Einstein's velocity addition formula, and it ensures that the resulting velocity never exceeds the speed of light.

2. Magnetism as a Relativistic Phenomenon

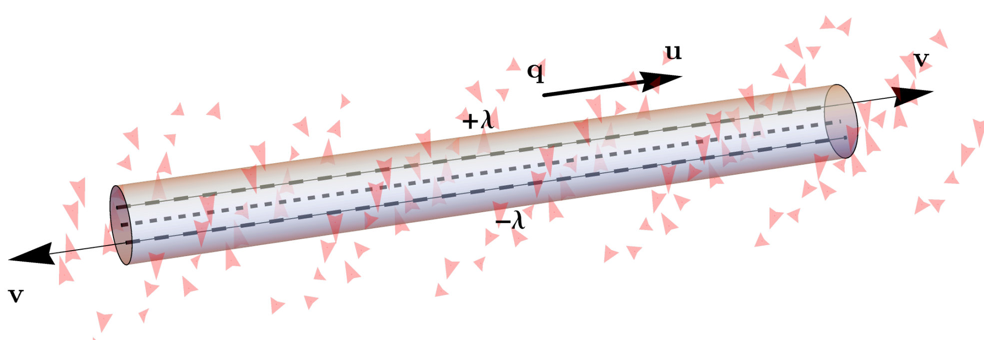

Figure 1: A moving charge q and line charges with linear charge density under the rested reference frame $\mathcal{S}$, red arrows indicate the magnetic field generated.

Let's consider a simple setup: suppose there's a line of charge with linear charge density λ, moving along the x-axis at a constant velocity $v\hat{x}$. Meanwhile, there's another line of charge with linear charge density $-\lambda$ on the exact same line, but moving towards the opposite direction with same speed v. Near the line of charge, there's a point charge q moving at velocity $u\hat{x}$.

Let's call the reference frame where we described the above setup $\mathcal{S}$. Figure 1 shows the setup under this reference frame.

- Under this reference frame, the two electric fields produced by the two line charges cancel each other out, leaving a net electric field $\boldsymbol{E} = \mathbf{0}$.
- Due to the motion of line charges, a current $\boldsymbol{I}$ is generated, given by:

$$\boldsymbol{I} = 2\lambda v\hat{x}$$

  and the magnetic field $\boldsymbol{B}$ produced by this current is given by:

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 \boldsymbol{I} \times \hat{r}}{2\pi\|\boldsymbol{r}\|} = \frac{\mu_0 \lambda v}{\pi\|\boldsymbol{r}\|}(\hat{x} \times \hat{r})$$

Therefore, the point charge q experiences a magnetic force given by:

$$\boldsymbol{F_B} = q\boldsymbol{u} \times \boldsymbol{B} = -\frac{\mu_0 \lambda v q u}{\pi\|\boldsymbol{r}\|}\hat{r} \tag{2}$$

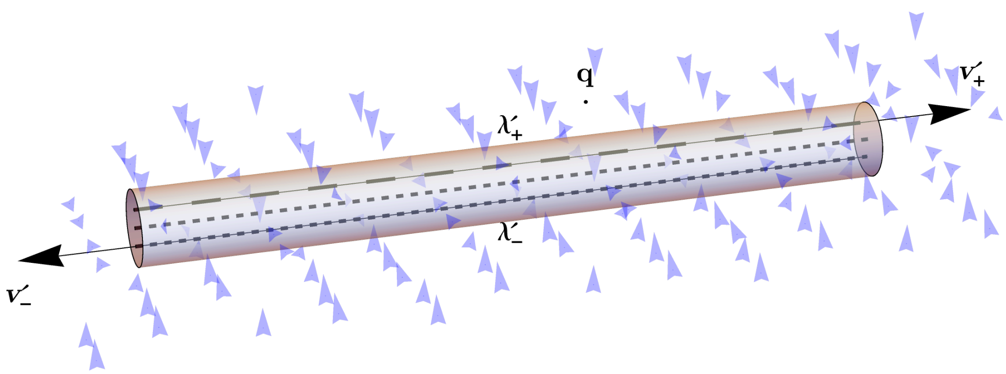

Figure 2: The same setup under the reference frame $\mathcal{S}'$, where the point charge q is at rest, blue arrows indicate the electric field generated.

Now, Consider this setup from another reference frame $\mathcal{S}'$, which is moving along the x-axis at a constant velocity $u\hat{x}$ relative to $\mathcal{S}$. Thus, the point charge q is at rest under this reference frame, and the two line charges are moving at:

$$v'_\pm = \frac{v \mp u}{1 \mp vu/c^2}$$

as given by Eq. 1.

This tells us that $v'_+ > v'_-$ in $\mathcal{S}'$, which accordingly causes the linear charge density to be different under $\mathcal{S}'$, which can be calculated using the Lorentz contraction as:

$$\lambda'_\pm = \pm\gamma_\pm\lambda_0 \quad \text{where } \gamma_\pm \equiv \frac{1}{\sqrt{1 - v'^2_\pm/c^2}}$$

It's worth noting here that $\lambda_0 \neq \lambda$ since the linear charges are already moving at speed v under $\mathcal{S}$, and thus we have:

$$\lambda = \gamma\lambda_0 \quad \text{where } \gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}$$

Notice that we can simplify $\gamma_\pm$:

$$\begin{aligned}
\gamma_\pm = \frac{1}{\sqrt{1 - v'^2_\pm/c^2}} &= \left[1 - c^{-2}\left(\frac{v \mp u}{1 \mp vu/c^2}\right)^2\right]^{-\frac{1}{2}} \\
&= \left[1 - c^2\left(\frac{v \mp u}{c^2 \mp vu}\right)^2\right]^{-\frac{1}{2}} \\
&= \left[\frac{\left(c^2 \mp vu\right)^2 - c^2(v \mp u)^2}{\left(c^2 \mp vu\right)^2}\right]^{-\frac{1}{2}} \\
&= \frac{c^2 \mp vu}{\sqrt{\left(c^2 \mp vu\right)^2 - c^2(v \mp u)^2}}
\end{aligned}$$

$$\boxed{\begin{aligned}
&\left(c^2 \mp vu\right)^2 - c^2(v \mp u)^2 \\
&= \left(c^4 + (vu)^2 \mp 2vuc^2\right) - \left(c^2v^2 + c^2u^2 \mp 2vuc^2\right) \\
&= c^4 - c^2v^2 - c^2u^2 + (vu)^2 \\
&= \left(c^2 - v^2\right)\left(c^2 - u^2\right)
\end{aligned}}$$

$$\begin{aligned}
&= \frac{c^2 \mp vu}{\sqrt{(c^2 - v^2)(c^2 - u^2)}} \\
&= \frac{1 \mp vu/c^2}{c^{-1}\sqrt{c^2 - v^2}\,c^{-1}\sqrt{c^2 - u^2}} \\
&= \frac{1}{\sqrt{1 - v^2/c^2}}\frac{1 \mp vu/c^2}{\sqrt{1 - u^2/c^2}} \\
&= \gamma\frac{1 \mp vu/c^2}{\sqrt{1 - u^2/c^2}}
\end{aligned}$$

Therefore, the total linear charge density under $\mathcal{S}'$ is given by:

$$\lambda'_{\text{total}} = \lambda_0(\gamma_+ - \gamma_-) = \gamma\lambda_0 \frac{-2vu/c^2}{\sqrt{1-u^2/c^2}} = \frac{-2\lambda vu}{c^2\sqrt{1-u^2/c^2}}$$

From Gauss's Law, the electric field produced by the line charges is given by:

$$\boldsymbol{E}'(\boldsymbol{r}) = \frac{\lambda'_{\text{total}}}{2\pi\varepsilon_0\|\boldsymbol{r}\|}\hat{r}$$

and the electric force experienced by the point charge q is given by:

$$\boldsymbol{F}'_{\boldsymbol{E}} = q\boldsymbol{E} = \frac{-2\lambda qvu}{c^2\sqrt{1-u^2/c^2}}\frac{\hat{r}}{2\pi\varepsilon_0\|\boldsymbol{r}\|} = -\frac{1}{\sqrt{1-u^2/c^2}}\frac{\lambda qvu}{\pi\|\boldsymbol{r}\|\varepsilon_0 c^2}\hat{r} \tag{3}$$

Now, we can compare this to the magnetic force $\boldsymbol{F}_{\boldsymbol{B}}$ under the reference frame $\mathcal{S}$.

Recall that $\boldsymbol{F}_{\boldsymbol{B}}$ is given by Eq. 2. We can transform $\boldsymbol{F}_{\boldsymbol{B}}$ under $\mathcal{S}$ to $\mathcal{S}'$ using the Lorentz transforms:

$$\boldsymbol{F}' = \gamma_u\boldsymbol{F}_{\boldsymbol{B}} = -\frac{1}{\sqrt{1-u^2/c^2}}\frac{\mu_0\lambda qvu}{\pi\|\boldsymbol{r}\|}\hat{r}$$

This still differs from Eq. 3. However, consider the fact that $\mu_0\varepsilon_0 c^2 = 1$, we can rewrite the above equation as:

$$\begin{aligned}\boldsymbol{F}' &= -\frac{1}{\sqrt{1-u^2/c^2}}\frac{(\varepsilon_0 c^2)^{-1}\lambda qvu}{\pi\|\boldsymbol{r}\|}\hat{r} \\ &= -\frac{1}{\sqrt{1-u^2/c^2}}\frac{\lambda qvu}{\pi\|\boldsymbol{r}\|\varepsilon_0 c^2}\hat{r}\end{aligned}$$

which is exactly the same as Eq. 3. Therefore, we can conclude that the magnetic force $\boldsymbol{F}_{\boldsymbol{B}}$ under $\mathcal{S}$ is exactly the same as the electric force $\boldsymbol{F}'_{\boldsymbol{E}}$ under $\mathcal{S}'$.

3. Transformation of Electromagnetic Fields

After examining a specific example to see how magnetism can be explained as a relativistic phenomenon, we would like to derive the general transformation rules for electromagnetic fields under different reference frames.

For this purpose, we can start by considering a system with two parallel, uniformly charged capacitor, respectively with charge density σ_0 and $-\sigma_0$, both at rest under the reference frame $\mathcal{S}_0$. This configuration is shown in Figure 3.

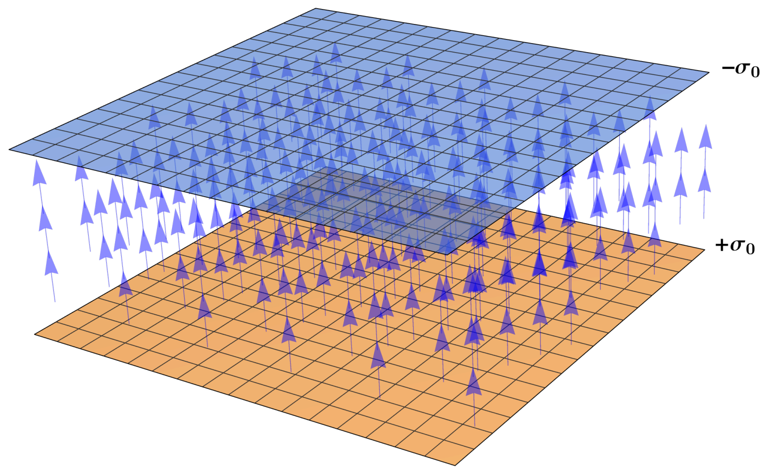

Figure 3: Two parallel, uniformly charged capacitor under the reference frame $\mathcal{S}_0$. Blue arrows indicate the electric field generated.

Since the two plates are at rest under $\mathcal{S}_0$, the electric field under $\mathcal{S}_0$ generated by the capacitor is given by:

$$\boldsymbol{E}_0 = \frac{\sigma_0}{\varepsilon_0}\hat{z}$$

Now, consider another reference frame $\mathcal{S}$, which is moving at a constant velocity $\boldsymbol{v}_0 = v_0\hat{x}$ relative to $\mathcal{S}_0$.

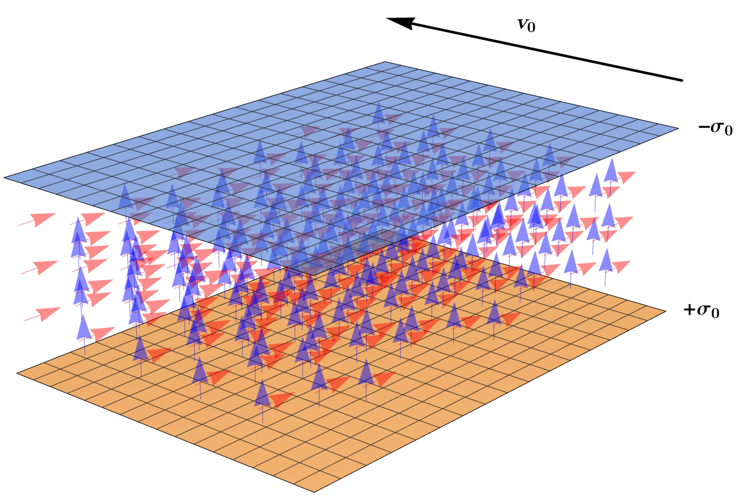

Figure 4: The same setup under the reference frame $\mathcal{S}$.

In this reference frame, the density of charges on the plates changes due to the Lorentz contraction (since length is contracted), and is given by:

$$\sigma = \gamma_0 \sigma_0 \quad \text{where } \gamma_0 \equiv \frac{1}{\sqrt{1 - v_0^2/c^2}}$$

The electric field under $\mathcal{S}$ is given by:

$$\boldsymbol{E} = \frac{\sigma}{\varepsilon_0}\hat{z} = \gamma_0 \frac{\sigma_0}{\varepsilon_0}\hat{z} = \gamma_0 \boldsymbol{E}_0 \tag{4}$$

This gives us the rule for transformation of electric fields under different reference frames:

Given an electric field $\boldsymbol{E}_0$ in a reference frame $\mathcal{S}_0$ without magnetic field, the electric field $\boldsymbol{E}$ in another reference frame $\mathcal{S}$ moving at a constant velocity $\boldsymbol{v}$ relative to $\mathcal{S}_0$ is:

$$\boldsymbol{E}^{\perp \boldsymbol{v}} = \gamma_0 \boldsymbol{E}_0^{\perp \boldsymbol{v}} \quad \text{and } \boldsymbol{E}^{\parallel \boldsymbol{v}} = \boldsymbol{E}_0^{\parallel \boldsymbol{v}} \tag{5}$$

where $\boldsymbol{E}^{\perp \boldsymbol{v}}$ and $\boldsymbol{E}^{\parallel \boldsymbol{v}}$ are the components of $\boldsymbol{E}$ perpendicular and parallel to $\boldsymbol{v}$.

Nevertheless, this isn't the end of the story. We would also like to know about how magnetic fields transform under different reference frames.

Since there's no moving charge in $\mathcal{S}_0$, there's no magnetic field induced. It would be helpful to start from a system with some initial magnetic field to see its rule of transform. Notice that $\mathcal{S}$ will be a good starting point since the charges are now moving, which becomes a current and can induce a magnetic field.

The magnetic field in $\mathcal{S}$ can easily calculated through Ampère's Law, and is given by:

$$\boldsymbol{B} = -\mu_0 \sigma v_0 \hat{y} \tag{6}$$

Now, let's consider a third reference frame $\overline{\mathcal{S}}$ that is moving at a constant velocity $\boldsymbol{v} = v\hat{x}$ relative to $\mathcal{S}$. Under this reference frame, the magnetic field is transformed but still exists, as well as the electric field.

Similarly, the velocity of this system relative to $\mathcal{S}_0$ can obtained by Eq. 1:

$$\overline{\boldsymbol{v}} = \frac{v_0 + v}{1 + v_0 v/c^2}\hat{x}$$

and the Lorentz gamma factor can be as well calculated:

$$\overline{\gamma} = \frac{1}{\sqrt{1 - \overline{v}^2/c^2}}$$

We can also get the charge density in $\overline{\mathcal{S}}$:

$$\overline{\sigma} = \overline{\gamma}\sigma_0$$

Those together give us the electric field $\overline{\boldsymbol{E}}$ and magnetic field $\overline{\boldsymbol{B}}$ in this reference frame:

$$\overline{\boldsymbol{E}} = \overline{\gamma}\boldsymbol{E}_0 = \left(\frac{\overline{\gamma}}{\gamma_0}\right)\frac{\sigma}{\varepsilon_0}\hat{z} \quad \text{and} \quad \overline{\boldsymbol{B}} = -\mu_0\overline{\sigma}\overline{v}\hat{y} = -\left(\frac{\overline{\gamma}}{\gamma_0}\right)\mu_0\sigma\overline{v}\hat{y}$$

We can simplify their common coefficient:

$$\frac{\overline{\gamma}}{\gamma_0} = \frac{\sqrt{1 - v_0^2/c^2}}{\sqrt{1 - \overline{v}^2/c^2}} = \sqrt{\frac{1 - v_0^2/c^2}{1 - c^{-2}\frac{(v_0+v)^2}{(1+v_0v/c^2)^2}}} = \sqrt{\frac{c^2 - v_0^2}{c^2 - \frac{(v_0+v)^2}{(1+v_0v/c^2)^2}}}$$

$$\boxed{\begin{aligned} c^2 - \frac{(v_0+v)^2}{(1+v_0v/c^2)^2} &= \frac{(c^2 + v_0v)(1 + v_0v/c^2) - (v_0+v)^2}{(1+v_0v/c^2)^2} \\ &= \frac{c^2 + \cancel{2v_0v} + (v_0v)^2/c^2 - v_0^2 - v^2 - \cancel{2v_0v}}{(1+v_0v/c^2)^2} \\ &= \frac{(c^2 - v_0^2) + v^2/c^2(v_0^2 - c^2)}{(1+v_0v/c^2)^2} \\ &= \frac{(c^2 - v_0)(1 - v^2/c^2)}{(1+v_0v/c^2)^2} \end{aligned}}$$

$$= \sqrt{\frac{\cancel{(c^2 - v_0^2)}(1 + v_0v/c^2)^2}{\cancel{(c^2 - v_0^2)}(1 - v^2/c^2)}} = \frac{1 + v_0v/c^2}{\sqrt{1 - v^2/c^2}} = \gamma\left(1 + \frac{v_0v}{c^2}\right)$$

where

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}}$$

Now, we can write $\overline{\boldsymbol{E}}$ and $\overline{\boldsymbol{B}}$ in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$ given by Eq. 4 and Eq. 6:

$$\overline{E}_z = \gamma\left(1 + \frac{v_0v}{c^2}\right)\frac{\sigma}{\varepsilon_0} = \gamma\left(E_y + v\frac{\sigma v_0}{c^2\varepsilon}\right) = \gamma(E_z - vB_y)$$

$\overline{\boldsymbol{B}}$ is a little more complicated:

$$\begin{aligned} \overline{B}_y = -\gamma\left(1 + \frac{v_0v}{c^2}\right)\mu_0\sigma\overline{v} &= -\gamma\mu_0\sigma\left(1 + \frac{v_0v}{c^2}\right)\left(\frac{v_0+v}{1+v_0v/c^2}\right) \\ &= -\gamma\mu_0\sigma\left(\frac{v_0+v}{1+v_0v/c^2} + \frac{v_0v}{c^2}\frac{v_0+v}{1+v_0v/c^2}\right) \end{aligned}$$

$$
\begin{aligned}
&= -\gamma\mu_0\sigma\left(\frac{v_0+v}{1+\frac{v_0 v}{c^2}}+\frac{v_0+v}{\frac{c^2}{v_0 v}+1}\right)\\
&= -\gamma\mu_0\sigma\frac{(v_0+v)\cancel{\left(1+\frac{v_0 v}{c^2}+1+\frac{c^2}{v_0 v}\right)}}{\cancel{\left(1+\frac{v_0 v}{c^2}\right)\left(1+\frac{c^2}{v_0 v}\right)}}\\
&= -\gamma\mu_0\sigma(v_0+v)\\
&= \gamma\left(B_y-\frac{\sigma}{\varepsilon_0 c^2}v\right)\\
&= \gamma\left(B_y-\frac{v}{c^2}E_z\right)
\end{aligned}
$$

This gives us the rules for transforming $\hat{y}$ component of magnetic fields and $\hat{z}$ component of electric fields:

$$\overline{E}_z=\gamma\big(E_z-vB_y\big)\quad\text{and}\quad\overline{B}_y=\gamma\left(B_y-\frac{v}{c^2}E_z\right)$$

We can also derive the rules for other components by aligning the capacitor in different directions. Suppose we align the capacitors parallel to the xz plane. This generates an electric field along $\hat{y}$ direction, and a magnetic field along $\hat{z}$ direction in $\mathcal{S}$:

$$\boldsymbol{E}=\frac{\sigma}{\varepsilon_0}\hat{y}\quad\text{and}\quad\boldsymbol{B}=\mu_0\sigma v_0\hat{z}$$

The transformed fields are similar, where we just need to replace $\hat{z}$ with $\hat{y}$ and $\hat{y}$ with $-\hat{z}$:

$$\overline{E}_y=\gamma\big(E_y+vB_z\big)\quad\text{and}\quad\overline{B}_z=\gamma\left(B_z+\frac{v}{c^2}E_y\right)$$

Eq. 5 already tells us about the $\hat{x}$ component of electric fields would be left unchanged:

$$\overline{E}_x=E_x$$

However, the magnetic fields in this case cancel out each other, producing a net magnetic field of 0. Therefore, we have to consider another setup to show the transformation for B_x.

Consider a solenoid where its axis is parallel to $\hat{x}$ direction. As we already know, the magnetic field inside the solenoid is given by:

$$\boldsymbol{B}=\mu_0 nI\hat{x}$$

where n is the number of loops per unit length and I is the current in the solenoid.

Now, in $\overline{\mathcal{S}}$, the current changes due to time dilation:

$$\overline{I} = \frac{\mathrm{d}Q}{\mathrm{d}\overline{t}} = \frac{1}{\gamma}\frac{\mathrm{d}Q}{\mathrm{d}t} = \frac{1}{\gamma}I$$

Meanwhile, the number of loops per unit length also changes due to the length contraction:

$$\overline{n} = \frac{\mathrm{d}N}{\mathrm{d}\overline{l}} = \gamma\frac{\mathrm{d}N}{\mathrm{d}l} = \gamma n$$

Those together show that the $\hat{x}$ component of magnetic field in $\overline{\mathcal{S}}$ is left unchanged:

$$\overline{B}_x = \mu_0\overline{n}\overline{I} = \mu_0 nI = B_x$$

We now have the complete rules for electromagnetic fields under different reference frames:

Given a reference frame $\overline{\mathcal{S}}$ moving relative to another reference frame $\mathcal{S}$ at a constant velocity $v\hat{x}$, the electromagnetic fields in $\overline{\mathcal{S}}$ can be calculated from the electromagnetic fields in $\mathcal{S}$ as:

$$\begin{aligned} \overline{E}_x &= E_x & \overline{B}_x &= B_x \\ \overline{E}_y &= \gamma(E_y + vB_z) & \overline{B}_z &= \gamma\left(B_z + \frac{v}{c^2}E_y\right) \\ \overline{E}_z &= \gamma(E_z - vB_y) & \overline{B}_y &= \gamma\left(B_y - \frac{v}{c^2}E_z\right) \end{aligned}$$

where

$$\gamma \equiv \frac{1}{\sqrt{1 - v^2/c^2}} \quad \text{is the Lorentz gamma factor}$$

4. Appendix: Mathematica Code for Figures

4.1. Figure 1

```
margin = 0.5;
length = 10;
extends = 3;
fieldRange = 3;
cylinderSize = 1;
Show[
 Graphics3D[{
   {Arrow[{{0, -length, margin}, {0, length + extends, margin}}]},
   {Thick, Dashing[{Medium, Medium}],
    Line[{{0, -length, margin}, {0, length, margin}}]},
```

```
  {Black,
   Text[Style["+\[Lambda]", Bold, 15,
     FontFamily -> "NewComputerModernMath"], {0, 0,
     cylinderSize + margin}]},
  {Black,
   Text[Style["v", Bold, 15,
     FontFamily -> "NewComputerModernMath"], {0, length + extends,
     cylinderSize + margin}]},
  {Arrow[{{0, length, -margin}, {0, -length - extends, -margin}}]},
  {Thick, Dashing[{Medium, Medium}],
   Line[{{0, -length, -margin}, {0, length, -margin}}]},
  {Black,
   Text[Style["-\[Lambda]", Bold, 15,
     FontFamily -> "NewComputerModernMath"], {0,
     0, -cylinderSize - margin}]},
  {Black,
   Text[Style["v", Bold, 15,
     FontFamily -> "NewComputerModernMath"], {0, -length -
      extends, -cylinderSize - margin}]},
  {Thick, Dashed, Line[{{0, -length, 0}, {0, length, 0}}]},
  {Opacity[.5],
   Cylinder[{{0, -length, 0}, {0, length, 0}}, cylinderSize]},
  {Thick, Arrow[{{0, 2, cylinderSize*2}, {0, 6, cylinderSize*2}}]},
  {Black,
   Text[Style["q", Bold, 15,
     FontFamily -> "NewComputerModernMath"], {0, 2,
     cylinderSize*2 + margin*1.5}]},
  {Black,
   Text[Style["u", Bold, 15,
     FontFamily -> "NewComputerModernMath"], {0, 6,
     cylinderSize*2 + margin*1.5}]},
  }],
VectorPlot3D[
 1/(2 Pi*(x^2 + z^2)^(1/2))*{z, 0, -x},
 {x, y, z} \[Element]
  Cylinder[{{0, -length - extends, 0}, {0, length + extends, 0}},
   fieldRange],
 RegionBoundaryStyle -> None,
 VectorStyle -> {Red, Opacity[.3]},
 VectorColorFunction -> None,
 VectorMarkers -> "Arrow",
 VectorSizes -> Tiny
 ],
Boxed -> False
]
```

4.2. Figure 2

```
margin = 0.5;
length = 10;
extends = 3;
fieldRange = 3;
cylinderSize = 1;
Show[
 Graphics3D[{
   {Arrow[{{0, -length, margin}, {0, length + extends, margin}}]},
   {Thick, Dashing[{Large, Large}],
    Line[{{0, -length, margin}, {0, length, margin}}]},
   {Black,
    Text[Style[
      "\!\(\*SubsuperscriptBox[\(\[Lambda]\), \(+\), \('\)]\)", Bold,
      15, FontFamily -> "NewComputerModernMath"], {0, 0,
      cylinderSize + margin}]},
   {Black,
    Text[Style["\!\(\*SubsuperscriptBox[\(v\), \(+\), \('\)]\)", Bold,
       15, FontFamily -> "NewComputerModernMath"], {0,
      length + extends, cylinderSize + margin}]},
   {Arrow[{{0, length, -margin}, {0, -length - extends, -margin}}]},
   {Thick, Dashing[{Small, Small}],
    Line[{{0, -length, -margin}, {0, length, -margin}}]},
   {Black,
    Text[Style[
      "\!\(\*SubsuperscriptBox[\(\[Lambda]\), \(-\), \('\)]\)", Bold,
      15, FontFamily -> "NewComputerModernMath"], {0,
      0, -cylinderSize - margin}]},
   {Black,
    Text[Style["\!\(\*SubsuperscriptBox[\(v\), \(-\), \('\)]\)", Bold,
       15, FontFamily -> "NewComputerModernMath"], {0, -length -
       extends, -cylinderSize - margin}]},
   {Thick, Dashed, Line[{{0, -length, 0}, {0, length, 0}}]},
   {Opacity[.5],
    Cylinder[{{0, -length, 0}, {0, length, 0}}, cylinderSize]},
   {Thick,
    Line[{{0, 2, cylinderSize*2}, {0, 2.01, cylinderSize*2}}]},
   {Black,
    Text[Style["q", Bold, 15,
      FontFamily -> "NewComputerModernMath"], {0, 2,
      cylinderSize*2 + margin*1.5}]}
   }],
 VectorPlot3D[
  -1/(2 Pi*(x^2 + z^2)^(1/2))*{x, 0, z},
  {x, y, z} \[Element]
```

```
   Cylinder[{{0, -length - extends, 0}, {0, length + extends, 0}},
    fieldRange],
  RegionBoundaryStyle -> None,
  VectorStyle -> {Blue, Opacity[.3]},
  VectorColorFunction -> None,
  VectorMarkers -> "Arrow",
  VectorSizes -> Tiny
  ],
 Boxed -> False
 ]
```

4.3. Figure 3

```
planeSideLength = 5;
planeHeight = 2;
margin = 0.5;
Show[
 Graphics3D[{
   {Black,
    Text[Style["+\!\(\*SubscriptBox[\(\[Sigma]\), \(0\)]\)", Bold, 15,
       FontFamily -> "NewComputerModernMath"], {planeSideLength +
       margin/2, planeSideLength + margin/2, 0}]},
   {Black,
    Text[Style["-\!\(\*SubscriptBox[\(\[Sigma]\), \(0\)]\)", Bold, 15,
       FontFamily -> "NewComputerModernMath"], {planeSideLength +
       margin/2, planeSideLength + margin/2, planeHeight}]}
   }],
 Plot3D[
  {0, planeHeight}, {x, 0, planeSideLength}, {y, 0, planeSideLength},
  PlotStyle -> {Opacity[.6]}
  ],
 VectorPlot3D[
  {0, 0, 2},
  {x, y, z} \[Element]
   Hexahedron[{{0, 0, margin}, {planeSideLength, 0,
      margin}, {planeSideLength, planeSideLength, margin}, {0,
      planeSideLength, margin}, {0, 0,
      planeHeight - margin}, {planeSideLength, 0,
      planeHeight - margin}, {planeSideLength, planeSideLength,
      planeHeight - margin}, {0, planeSideLength,
      planeHeight - margin}}],
  RegionBoundaryStyle -> None,
  VectorStyle -> {Blue, Opacity[.45]},
  VectorColorFunction -> None,
  VectorMarkers -> "Arrow",
  VectorSizes -> Small
```

```
  ],
 Boxed -> False
 ]
```

4.4. Figure 4

```
planeSideLength = 5;
planeSideLengthContracted = 3.5;
planeHeight = 2;
margin = 0.5;
region =
  Hexahedron[{{0, 0, margin}, {planeSideLengthContracted, 0,
     margin}, {planeSideLengthContracted, planeSideLength,
     margin}, {0, planeSideLength, margin}, {0, 0,
     planeHeight - margin}, {planeSideLengthContracted, 0,
     planeHeight - margin}, {planeSideLengthContracted,
     planeSideLength, planeHeight - margin}, {0, planeSideLength,
     planeHeight - margin}}];
Show[
 Graphics3D[{
   {Black,
    Text[Style["+\!\(\*SubscriptBox[\(\[Sigma]\), \(0\)]\)", Bold, 15,
       FontFamily ->
       "NewComputerModernMath"], {planeSideLengthContracted +
       margin/2, planeSideLength + margin/2, 0}]},
   {Black,
    Text[Style["-\!\(\*SubscriptBox[\(\[Sigma]\), \(0\)]\)", Bold, 15,
       FontFamily ->
       "NewComputerModernMath"], {planeSideLengthContracted +
       margin/2, planeSideLength + margin/2, planeHeight}]},
   {Thick,
    Arrow[{{planeSideLengthContracted, planeSideLength,
       planeHeight + margin}, {0, planeSideLength,
       planeHeight + margin}}]},
   {Black,
    Text[Style["\!\(\*SubscriptBox[\(v\), \(0\)]\)", Bold, 15,
      FontFamily ->
       "NewComputerModernMath"], {planeSideLengthContracted/2,
      planeSideLength, planeHeight + margin*1.5}]}
   }],
 Plot3D[
  {0, planeHeight}, {x, 0, planeSideLengthContracted}, {y, 0,
   planeSideLength},
  PlotStyle -> {Opacity[.6]}
  ],
 VectorPlot3D[
```

```
  {0, 0, 4},
  {x, y, z} \[Element]
   TransformedRegion[region, TranslationTransform[{-.08, -.08, 0}]],
  RegionBoundaryStyle -> None,
  VectorStyle -> {Blue, Opacity[.45]},
  VectorColorFunction -> None,
  VectorMarkers -> "Arrow",
  VectorSizes -> Small
  ],
 VectorPlot3D[
  {0, 2, 0},
  {x, y, z} \[Element] region,
  RegionBoundaryStyle -> None,
  VectorStyle -> {Red, Opacity[.45]},
  VectorColorFunction -> None,
  VectorMarkers -> "Arrow",
  VectorSizes -> Small
  ],
 Boxed -> False
 ]
```

Bibliography

[1] D. Griffiths, *Introduction to Electrodynamics* (Pearson Education, 2014).