1. Inverse of Matrix

In this section, we're going to prove a way to determine whether a matrix is invertible without calculating its inverse. We are also going to derive an explicit formula for the inverse of a matrix.

1.1. Basic Definitions

Recall definitions of determinant and matrix minors:

Definition. The matrix minor of a $n \times n$ matrix $\mathbf{A}$ is a $(n-1) \times (n-1)$ matrix that remove the ith row and jth column of the matrix $\mathbf{A}$, denoted as $\mathbf{M}_{ij}$.

In this paper, whenever it's not ambiguous, I'll directly use $\mathbf{M}$ to denote the matrix minor of a matrix. When there's possible ambiguity, I'll use $\mathbf{M}_\mathbf{A}$ for minor of $\mathbf{A}$.

It's worth noting that since matrix minors are constructed by removing a row and a column, the following stands true:

Theorem. For every matrix minor $\mathbf{M}_{\mathbf{A},ij}$ of a $n \times n$ matrix $\mathbf{A}$, it's always possible to construct a $n \times n$ matrix $\mathbf{B}$ so that it has the same matrix minor $\mathbf{M}_{\mathbf{B},ij} = \mathbf{M}_{\mathbf{A},ij}$.

This theorem is quite trivial so we won't prove it here.

Definition. The determinant of a $n \times n$ matrix $\mathbf{A}$ is defined as:

$$\det \mathbf{A} := \begin{cases} \sum_{j=1}^{n} (-1)^{1+j} \mathbf{A}_{1j} \det \mathbf{M}_{\mathbf{1j}} & \text{if } n \geq 1 \\ \mathbf{A}_{11} & \text{if } n = 1 \end{cases}$$

1.2. Cofactor Matrix

We can begin with the concept of a **cofactor matrix**:

Definition. For a $n \times n$ matrix $\mathbf{A}$, a cofactor matrix $\mathbf{C}$ is a $n \times n$ matrix, each entry defined as:

$$\mathbf{C}_{ij} = (-1)^{i+j} \cdot \det \mathbf{M}_{ij} \tag{1}$$

With cofactors defined, determinant also be written as:

$$\det \mathbf{A} = \sum_{j=1}^{n} \mathbf{A}_{ij} \cdot \mathbf{C}_{ij} \quad \text{for any } 1 \leq i \leq n \tag{2}$$

1.3. Properties of Cofactor Matrix

To proceed, we can first prove some useful properties of the cofactor matrix.

It's easy to see that the transpose of the cofactor matrix can be expressed as:

$$\mathbf{C}^{\mathrm{T}}_{ij} = (-1)^{i+j} \cdot \det \mathbf{M}_{ji}$$

This is also known as the **adjugate matrix** of $\mathbf{A}$, denoted as $\operatorname{adj}(\mathbf{A})$.

Theorem. For a $n \times n$ matrix $\mathbf{A}$, the following holds:

$$\mathbf{A} \operatorname{adj}(\mathbf{A}) = \det \mathbf{A} \cdot \mathbf{I}$$

Proof. Consider each entry of the matrix multiplication:

$$(\mathbf{A} \operatorname{adj}(\mathbf{A}))_{ij} = \left(\mathbf{A}\mathbf{C}^{\mathrm{T}}\right)_{ij} = \sum_{k=1}^{n} \mathbf{A}_{ik} \cdot \mathbf{C}^{\mathrm{T}}_{kj}$$

Here, we can split the problem into two cases:
1. If $i = j$, the entry is simply $\det(\mathbf{A})$:

$$\begin{aligned} \left(\mathbf{A}\mathbf{C}^{\mathrm{T}}\right)_{ii} &= \sum_{k=1}^{n} \mathbf{A}_{ik} \cdot \mathbf{C}^{\mathrm{T}}_{ki} \\ &= \sum_{k=1}^{n} \mathbf{A}_{ik} \cdot \mathbf{C}_{ik} \quad &\text{(by def. of transpose)} \\ &= \det(\mathbf{A}) \quad &\text{(by Eq. 2)} \end{aligned}$$

Before proving the next part, we'll first prove a lemma:

Lemma. If a $n \times n$ matrix $\mathbf{A}$ has two identical rows, then $\det \mathbf{A} = 0$.

Proof. Recall the property of determinant that exchanging two rows of $\mathbf{A}$ reverse $\det \mathbf{A}$ to $-\det \mathbf{A}$. If $\mathbf{A}$ has two identical rows, exchanging them won't change the matrix, which means $\det \mathbf{A} = -\det \mathbf{A}$. Therefore, $\det \mathbf{A} = 0$. ■

Now, with the lemma proven, we can continue with the second case:

2. If $i \neq j$, we want to show that the entry is 0:
Consider another $n \times n$ matrix $\mathbf{B}$ defined as:

$$\mathbf{B}_{pq} := \begin{cases} \mathbf{A}_{pq} & \text{if } p \neq j \\ \mathbf{A}_{iq} & \text{if } p = j \end{cases}$$

which is basically replacing the jth row of $\mathbf{A}$ with its ith row.

Since it has the same entries as $\mathbf{A}$ everywhere except at the jth row, we have:

$$\mathbf{M}_{\mathbf{B}} = \mathbf{M}$$

Since the jth row of $\mathbf{B}$ is same as ith row of $\mathbf{A}$, $\left(\mathbf{A}\mathbf{C}^{\mathrm{T}}\right)_{ij}$ can be written as:

$$\begin{aligned} \left(\mathbf{A}\mathbf{C}^{\mathrm{T}}\right)_{ij} &= \sum_{k=1}^{n} \mathbf{A}_{ik} \cdot \mathbf{C}^{\mathrm{T}}_{kj} \\ &= \sum_{k=1}^{n} \mathbf{A}_{ik} \cdot (-1)^{k+j} \cdot \det \mathbf{M}_{jk} \\ &= \sum_{k=1}^{n} \mathbf{B}_{jk} \cdot (-1)^{k+j} \cdot \det \mathbf{M}_{jk} \\ &= \sum_{k=1}^{n} \mathbf{B}_{jk} \cdot \mathbf{C}^{\mathrm{T}}_{kj} \\ &= \sum_{k=1}^{n} \mathbf{B}_{jk} \cdot \mathbf{C}_{jk} \\ &= \det(\mathbf{B}) \end{aligned}$$

However, by the lemma we proved, since ith and jth row of $\mathbf{B}$ are the same, we have $\det \mathbf{B} = 0$, which means:

$$\left(\mathbf{A}\mathbf{C}^{\mathrm{T}}\right)_{ij} = 0$$

Now, we have proved that for both cases, which means:

$$\begin{aligned} (\mathbf{A}\,\mathrm{adj}(\mathbf{A}))_{ij} &= \begin{cases} \det \mathbf{A} & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \\ &= \det \mathbf{A} \cdot \mathbf{I}_{ij} \end{aligned}$$

Therefore,

$$\mathbf{A}\,\mathrm{adj}(\mathbf{A}) = \det \mathbf{A} \cdot \mathbf{I} \qquad \blacksquare$$

1.4. Existence of Inverse Matrix with Determinant

Theorem. for a $n \times n$ matrix $\mathbf{A}$, the following relation holds:

$$\det \mathbf{A} \neq 0 \quad \Longleftrightarrow \quad \exists A^{-1}$$

With the theorem we proved above, it's quite easy to prove this:

Proof. Since the theorem is both way, we first prove the forward direction:

1. Assume $\det \mathbf{A} \neq 0$, prove $\mathbf{A}^{-1}$ exists:

$$\begin{aligned} \mathbf{A}\,\mathrm{adj}(\mathbf{A}) &= \det \mathbf{A} \cdot \mathbf{I} && \text{(from previous theorem)} \\ \mathbf{I} &= \mathbf{A}\left(\frac{1}{\det \mathbf{A}}\right)\mathrm{adj}(\mathbf{A}) && (\text{since} \det \mathbf{A} \neq 0\ ) \\ \mathbf{A}^{-1} &= \frac{1}{\det \mathbf{A}} \cdot \mathrm{adj}(\mathbf{A}) \end{aligned}$$

2. Assume $\mathbf{A}^{-1}$ exists, prove $\det(\mathbf{A}) \neq 0$:

$$\begin{aligned} \mathbf{A}\mathbf{A}^{-1} &= \mathbf{I} \\ \det\left(\mathbf{A}\mathbf{A}^{-1}\right) &= \det \mathbf{I} \\ \det \mathbf{A} \cdot \det \mathbf{A}^{-1} &= 1 && \text{(property of determinant)} \end{aligned}$$

Since $\det(\mathbf{A})$ multiple by some other number is 1, it must be non-zero. Therefore, we have:

$$\det \mathbf{A} \neq 0$$

■

2. Cramer's Rule

Theorem. (Cramer's Rule) If $\det \mathbf{A} \neq 0$, given a linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$,

$$x_j = \frac{\det \mathbf{B_j}}{\det \mathbf{A}} \quad \text{where } \left(\mathbf{B_j}\right)_{ik} = \begin{cases} A_{ik} & \text{if } k \neq j \\ \boldsymbol{b}_i & \text{if } k = j \end{cases}$$

gives the unique solution $\boldsymbol{x}$ to the linear system.

Proof. Since $\det \mathbf{A} \neq 0$, we know that $\mathbf{A}^{-1}$ exists, and is given by:

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \cdot \mathrm{adj}(\mathbf{A})$$

Therefore, the solution to the linear system is:

$$\boldsymbol{x} = \mathbf{A}^{-1}\boldsymbol{b} = \frac{1}{\det \mathbf{A}} \cdot \operatorname{adj}(\mathbf{A})\boldsymbol{b}$$

$$x_j = \frac{(j \text{ th row of } \operatorname{adj}(\mathbf{A})) \cdot \boldsymbol{b}}{\det \mathbf{A}}$$

Notice the similarity of this equation with the equation in Cramer's Rule. Our target is to prove that:

$$(j\text{th row of } \operatorname{adj}(\mathbf{A})) \cdot \boldsymbol{b} = \det \mathbf{B_j} \tag{3}$$

We can start by considering equivalence form of vector dot product. It's quite obvious that $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{b}$. Therefore, we have:

$$\begin{aligned}\text{lhs of Eq. 3} &= (j\text{th row of } \operatorname{adj}(\mathbf{A}))^{\mathrm{T}}\boldsymbol{b} \\ &= (j\text{th column of } \mathbf{C}) \cdot \boldsymbol{b}\end{aligned}$$

Next, consider the other side with cofactor formula of determinant along the jth column. Since $\mathbf{B_j}$ only replaced the jth column of $\mathbf{A}$, its cofactor matrix remains the same.

$$\begin{aligned}\text{lhs of Eq. 3} &= \det \mathbf{B_j} \\ &= \sum_{i=1}^{n} \left(\mathbf{B_j}\right)_{ij} \mathbf{C}_{ij} \\ &= \sum_{i=1}^{n} b_i \cdot \mathbf{C}_{ij} && (\text{by def. of } \mathbf{B_j}\ ) \\ &= (j\text{th column of } \mathbf{C}) \cdot \boldsymbol{b}\end{aligned}$$

Thus, we have lhs = rhs. Cramer's rule is proved. ■

2.1. An Alternative Approach

Another quite intuitive proof was provided on the textbook. It felt neat to me so I'm also going to share it here.

Proof. Since we know that $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$, let's consider the multiplication of which two matrix would produce $\mathbf{B_j}$.

We know that $\mathbf{AI} = \mathbf{A}$, and matrix multiplication can be considered as taking each column of the right matrix and applying the left matrix to it, which means:

$$\mathbf{AI} = \begin{pmatrix} | & | & | & & | \\ \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \dots & \boldsymbol{v_n} \\ | & | & | & & | \end{pmatrix} \quad \text{where } \boldsymbol{v_k} \text{ are cols of } \mathbf{A}$$

If we replace one column of $\mathbf{I}$ with $\boldsymbol{x}$, call it $\mathbf{X_j}$, the resulting matrix would be $\mathbf{B_j}$ since we know that applying $\mathbf{A}$ send $\boldsymbol{x}$ to $\boldsymbol{b}$:

$$\mathbf{AX_j} = \begin{pmatrix} | & | & | & & | & & | \\ \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \dots & \boldsymbol{x} & \dots & \boldsymbol{v_n} \\ | & | & | & & | & & | \end{pmatrix} = \mathbf{B_j}$$

With the property of determinant, taking determinant on both sides gives:

$$\det(\mathbf{AX_j}) = \det(\mathbf{B_j})$$

$$\det \mathbf{A} \cdot \det \mathbf{X_j} = \det \mathbf{B_j}$$

Since the matrix minor after removing the jth column of $\mathbf{X_j}$ was just the identity matrix, its determinant $\det \mathbf{X_j} = \boldsymbol{x}_j$:

$$\det \mathbf{A} \cdot \boldsymbol{x}_j = \det \mathbf{B_j}$$

$$\boldsymbol{x}_j = \frac{\det \mathbf{B_j}}{\det \mathbf{A}} \qquad \blacksquare$$