Inverse of Matrix & Cramer's Rule

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1. Inverse of Matrix

In this section, we're going to prove a way to determine whether a matrix is invertible without calculating its inverse. We are also going to derive an explicit formula for the inverse of a matrix.

1.1. Basic Definitions

Definition. The matrix minor of a $n \times n$ matrix $\mathbf{A}$ is a $(n-1) \times (n-1)$ matrix that remove the ith row and jth column of the matrix $\mathbf{A}$, denoted as $\mathbf{M}_{ij}$.

Definition. The determinant of a $n \times n$ matrix $\mathbf{A}$ is defined as:

$$\det \mathbf{A} := \begin{cases} \sum_{j=1}^{n} (-1)^{1+j} \mathbf{A}_{1j} \det \mathbf{M}_{\mathbf{1j}} & \text{if } n \geq 1 \\ \mathbf{A}_{11} & \text{if } n = 1 \end{cases}$$

1.2. Cofactor Matrix

Definition. For a $n \times n$ matrix $\mathbf{A}$, a cofactor matrix $\mathbf{C}$ is a $n \times n$ matrix, each entry defined as:

$$\mathbf{C}_{ij} = (-1)^{i+j} \cdot \det \mathbf{M}_{ij} \tag{1}$$

With cofactors defined, determinant also be written as:

$$\det \mathbf{A} = \sum_{j=1}^{n} \mathbf{A}_{ij} \cdot \mathbf{C}_{ij} \quad \text{for any } 1 \leq i \leq n \tag{2}$$

1.3. Properties of Cofactor Matrix

The transpose of the cofactor matrix can be expressed as:

$$\mathbf{C}^{\mathrm{T}}_{ij} = (-1)^{i+j} \cdot \det \mathbf{M}_{ji}$$

We calls this the **adjugate matrix** of $\mathbf{A}$, denoted as $\mathrm{adj}(\mathbf{A})$.

Before proving the next part, we'll first prove a lemma:

Lemma. If a $n \times n$ matrix $\mathbf{A}$ has two identical rows, then $\det \mathbf{A} = 0$.

Proof. Recall the property of determinant that exchanging two rows of $\mathbf{A}$ reverse $\det \mathbf{A}$ to $-\det \mathbf{A}$. If $\mathbf{A}$ has two identical rows, exchanging them won't change the matrix, which means $\det \mathbf{A} = -\det \mathbf{A}$. Therefore, $\det \mathbf{A} = 0$. $\blacksquare$

Let's prove another theorem:

Theorem. For a $n \times n$ matrix $\mathbf{A}$, the following holds:

$$\mathbf{A}\,\mathrm{adj}(\mathbf{A}) = \det \mathbf{A} \cdot \mathbf{I}$$

Proof. Consider each entry of the matrix multiplication:

$$(\mathbf{A}\,\mathrm{adj}(\mathbf{A}))_{ij} = \left(\mathbf{A}\mathbf{C}^{\mathrm{T}}\right)_{ij} = \sum_{k=1}^{n} \mathbf{A}_{ik} \cdot \mathbf{C}^{\mathrm{T}}_{kj}$$

Here, we can split the problem into two cases:

1. If $i = j$:

$$\begin{aligned}
\left(\mathbf{A}\mathbf{C}^{\mathrm{T}}\right)_{ii} &= \sum_{k=1}^{n} \mathbf{A}_{ik} \cdot \mathbf{C}^{\mathrm{T}}_{ki} \\
&= \sum_{k=1}^{n} \mathbf{A}_{ik} \cdot \mathbf{C}_{ik} \quad \text{(by def. of transpose)} \\
&= \det(\mathbf{A}) \qquad \text{(by Eq. 2)}
\end{aligned}$$

2. If $i \neq j$:
Consider another $n \times n$ matrix $\mathbf{B}$ defined as:

$$\mathbf{B}_{pq} := \begin{cases} \mathbf{A}_{pq} & \text{if } p \neq j \\ \mathbf{A}_{iq} & \text{if } p = j \end{cases}$$

which is basically replacing the jth row of $\mathbf{A}$ with its ith row.

Since it has the same entries as $\mathbf{A}$ everywhere except at the jth row, we have:

$$\mathbf{M}_{\mathbf{B}} = \mathbf{M}$$

$$
\begin{aligned}
\left(\mathbf{A}\mathbf{C}^{\mathrm{T}}\right)_{ij} &= \sum_{k=1}^{n} \mathbf{A}_{ik} \cdot \mathbf{C}^{\mathrm{T}}_{kj} \\
&= \sum_{k=1}^{n} \mathbf{A}_{ik} \cdot (-1)^{k+j} \cdot \det \mathbf{M}_{jk} \\
&= \sum_{k=1}^{n} \mathbf{B}_{jk} \cdot (-1)^{k+j} \cdot \det \mathbf{M}_{jk} \\
&= \sum_{k=1}^{n} \mathbf{B}_{jk} \cdot \mathbf{C}^{\mathrm{T}}_{kj} \\
&= \det(\mathbf{B})
\end{aligned}
$$

However, by the lemma we proved, since ith and jth row of $\mathbf{B}$ are the same, we have $\det \mathbf{B} = 0$, which means:

$$\left(\mathbf{A}\mathbf{C}^{\mathrm{T}}\right)_{ij} = 0$$

Now, we have proved that for both cases, which means:

$$\begin{aligned}(\mathbf{A}\,\mathrm{adj}(\mathbf{A}))_{ij} &= \begin{cases}\det \mathbf{A} & \text{if } i = j \\ 0 & \text{if } i \neq j\end{cases} \\ &= \det \mathbf{A} \cdot \mathbf{I}_{ij}\end{aligned}$$

Therefore,

$$\mathbf{A}\,\mathrm{adj}(\mathbf{A}) = \det \mathbf{A} \cdot \mathbf{I}$$

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1.4. Existence of Inverse Matrix with Determinant

Theorem. for a $n \times n$ matrix $\mathbf{A}$, the following relation holds:

$$\det \mathbf{A} \neq 0 \quad \Longleftrightarrow \quad \exists A^{-1}$$

Proof. Since the theorem is both way, we first prove the forward direction:

1. Assume $\det \mathbf{A} \neq 0$, prove $\mathbf{A}^{-1}$ exists:

$$\mathbf{A}\,\mathrm{adj}(\mathbf{A}) = \det \mathbf{A} \cdot \mathbf{I} \qquad \text{(from previous theorem)}$$

$$\mathbf{I} = \mathbf{A}\left(\frac{1}{\det \mathbf{A}}\right)\mathrm{adj}(\mathbf{A}) \qquad (\text{since} \det \mathbf{A} \neq 0\ )$$

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \cdot \mathrm{adj}(\mathbf{A})$$

2. Assume $\mathbf{A}^{-1}$ exists, prove $\det(\mathbf{A}) \neq 0$:

$$\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$$

$$\det\left(\mathbf{A}\mathbf{A}^{-1}\right) = \det \mathbf{I}$$

$$\det \mathbf{A} \cdot \det \mathbf{A}^{-1} = 1 \qquad \text{(property of determinant)}$$

Since $\det(\mathbf{A})$ multiple by some other number is 1, it must be non-zero. Therefore, we have:

$$\det \mathbf{A} \neq 0$$

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2. Cramer's Rule

Theorem. (Cramer's Rule) If $\det \mathbf{A} \neq 0$, given a linear system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$,

$$x_j = \frac{\det \mathbf{B_j}}{\det \mathbf{A}} \quad \text{where } \left(\mathbf{B_j}\right)_{ik} = \begin{cases} A_{ik} & \text{if } k \neq j \\ \boldsymbol{b}_i & \text{if } k = j \end{cases}$$

gives the unique solution $\boldsymbol{x}$ to the linear system.

Proof. Since $\det \mathbf{A} \neq 0$, we know that $\mathbf{A}^{-1}$ exists, and is given by:

$$\mathbf{A}^{-1} = \frac{1}{\det \mathbf{A}} \cdot \operatorname{adj}(\mathbf{A})$$

Therefore, the solution to the linear system is:

$$\boldsymbol{x} = \mathbf{A}^{-1}\boldsymbol{b} = \frac{1}{\det \mathbf{A}} \cdot \operatorname{adj}(\mathbf{A})\boldsymbol{b}$$

$$x_j = \frac{(j \text{ th row of } \operatorname{adj}(\mathbf{A})) \cdot \boldsymbol{b}}{\det \mathbf{A}}$$

Notice the similarity of this equation with the equation in Cramer's Rule. Our target is to prove that:

$$(j\text{th row of } \operatorname{adj}(\mathbf{A})) \cdot \boldsymbol{b} = \det \mathbf{B_j} \tag{3}$$

We can start by considering equivalence form of vector dot product. It's quite obvious that $\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{a}^{\mathrm{T}} \boldsymbol{b}$. Therefore, we have:

$$\begin{aligned} \text{lhs of Eq. 3} &= (j\text{th row of } \operatorname{adj}(\mathbf{A}))^{\mathrm{T}} \boldsymbol{b} \\ &= (j\text{th column of } \mathbf{C}) \cdot \boldsymbol{b} \end{aligned}$$

Next, consider the other side with cofactor formula of determinant along the jth column. Since $\mathbf{B_j}$ only replaced the jth column of $\mathbf{A}$, its cofactor matrix remains the same.

$$\begin{aligned}
\text{lhs of Eq. 3} &= \det \mathbf{B_j} \\
&= \sum_{i=1}^{n} \left(\mathbf{B_j}\right)_{ij} \mathbf{C}_{ij} \\
&= \sum_{i=1}^{n} b_i \cdot \mathbf{C}_{ij} && (\text{by def. of } \mathbf{B_j}\ ) \\
&= (j\text{th column of } \mathbf{C}) \cdot \boldsymbol{b}
\end{aligned}$$

Thus, we have lhs = rhs. Cramer's rule is proved. ∎