Prove **Green's Identities**:

$$\oiint_{\partial W} f\boldsymbol{\nabla} g \cdot \hat{\boldsymbol{n}} \,\mathrm{d}S = \iiint_W (f\Delta g + \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g) \,\mathrm{d}V$$

and

$$\oiint_{\partial W} (f\boldsymbol{\nabla} g - g\boldsymbol{\nabla} f) \cdot \hat{\boldsymbol{n}} \,\mathrm{d}S = \iiint_W (f\Delta g - g\Delta f) \,\mathrm{d}V$$

Proof:

First, consider **Ostrogradsky's Theorem**:

$$\oiint_{\partial W} \boldsymbol{F} \cdot \hat{\boldsymbol{n}} \,\mathrm{d}S = \iiint_W \boldsymbol{\nabla} \cdot \boldsymbol{F} \,\mathrm{d}V$$

Applying it to the LHS of the first identity (where $\boldsymbol{F} = f\boldsymbol{\nabla} g$), we have:

$$\begin{aligned}\oiint_{\partial W} f\boldsymbol{\nabla} g \cdot \hat{\boldsymbol{n}} \,\mathrm{d}S &= \iiint_W \boldsymbol{\nabla} \cdot (f\boldsymbol{\nabla} g) \,\mathrm{d}V \\ &= \iiint_W (f\boldsymbol{\nabla}^2 g + \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g) \,\mathrm{d}V \\ &= \iiint_W (f\Delta g + \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g) \,\mathrm{d}V\end{aligned}$$

which completes the proof of the first identity.

Now, we try to prove the second identity. Notice that we can split the LHS of the second identity into two parts:

$$\oiint_{\partial W} (f\boldsymbol{\nabla} g - g\boldsymbol{\nabla} f) \cdot \hat{\boldsymbol{n}} \,\mathrm{d}S = \oiint_{\partial W} (f\boldsymbol{\nabla} g \cdot \hat{\boldsymbol{n}} - g\boldsymbol{\nabla} f \cdot \hat{\boldsymbol{n}}) \,\mathrm{d}S$$

Next, we can simply apply the first identity to each of the two parts:

$$\begin{aligned}\text{RHS} &= \iiint_W (f\Delta g + \boldsymbol{\nabla} f \cdot \boldsymbol{\nabla} g - g\Delta f - \boldsymbol{\nabla} g \cdot \boldsymbol{\nabla} f) \,\mathrm{d}V \\ &= \iiint_W (f\Delta g - g\Delta f) \,\mathrm{d}V\end{aligned}$$

which completes the proof of the second identity.