$$
\begin{aligned}
\iiint_S x^2(-z) - y^2 z \,\mathrm{d}V &= \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{\sqrt{1-z^2}} -r\left(zr^2\cos^2\theta + zr^2\sin^2\theta\right) \,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta \\
&= \int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{\sqrt{1-z^2}} -r\left(zr^2\right) \,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta \\
&= -\int_0^{\frac{\pi}{2}} \int_0^1 \int_0^{\sqrt{1-z^2}} zr^3 \,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}\theta \\
&= -\int_0^{\frac{\pi}{2}} \int_0^1 \frac{1}{4}zr^4 \Big|_{r=0}^{\sqrt{1-z^2}} \,\mathrm{d}z\,\mathrm{d}\theta \\
&= -\frac{1}{4}\int_0^{\frac{\pi}{2}} \int_0^1 z\left(1-z^2\right)^2 \,\mathrm{d}z\,\mathrm{d}\theta \\
&= -\frac{1}{4}\int_0^{\frac{\pi}{2}} -\frac{1}{6}\left(1-z^2\right)^3 \Big|_{z=0}^{1} \,\mathrm{d}\theta \\
&= \frac{1}{24}\int_0^{\frac{\pi}{2}} -1 \,\mathrm{d}\theta \\
&= \frac{1}{24}\cdot -\frac{\pi}{2} \\
&= -\frac{\pi}{48}
\end{aligned}
$$